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Fourier Transforms of Smooth Functions on Certain Nilpotent Lie Groups*

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The authors consider irreducible representations $\pi \in \hat{N}$ of a nilpotent Lie group and define a Fourier transform for Schwartz class (and other) functions ϕ on N by forming the kernels $K_\phi(x, y)$ of the trace class operations $\pi_\phi = \int_N \phi(n)\pi_n \, dn$, regarding the π as modeled in $L^2(\mathbf{R}^k)$ for all π in general position. For a special class of groups they show that the models, and parameters λ labeling the representations in general position, can be chosen so the joint behavior of the kernels $K_\phi(x, y, \lambda)$ can be interpreted in a useful way. The variables (x, y, λ) run through a Zariski open set in $\mathbf{R}^n$, $n = \dim N$. The authors show there is a polynomial map $u = A(x, y, \lambda)$ that is a birational isomorphism $A: \mathbf{R}^n \to \mathbf{R}^n$ with the following properties. The Fourier transforms $F_1\phi = K_\phi(x, y, \lambda)$ all factor through A to give "rationalized" Fourier transforms $F\phi(u)$ such that $F\phi \circ A = F_1\phi$. On the rationalized parameter space a function $f(u)$ is of the form $F_\phi = f \Leftrightarrow f$ is Schwartz class on R^n. If polynomial operators $T \in P(N)$ are transferred to operators $\tilde{T}$ on $\mathbf{R}^n$ such that $F(T\phi) = \tilde{T}(F\phi)$, $P(N)$ is transformed isomorphically to $P(\mathbf{R}^n)$.

1. INTRODUCTION

Let N be a connected, simply connected n-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$. Kirillov, in his thesis [6], showed that the irreducible unitary representations of N are parametrized by the orbits of the coadjoint action of N on $\mathfrak{n}^*$. The parametrization ties the orbit to the representation in two ways: there is a procedure for computing the representation $\pi_\mathcal{O}$ associated with an

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orbit $\mathcal{O} \subseteq \mathfrak{n}^*$, and the trace character of π (a tempered distribution on $\mathfrak{n}$) is intimately connected with the orbit. For future reference we give the latter result. Let φ be a Schwartz class function on N, and let $\psi = \varphi \circ \exp \in \mathscr{S}(\mathfrak{n})$; then $\pi_\varphi = \int_N \varphi(n)\, \pi_n\, dn$ is a trace class operator. The Euclidean Fourier transform $\mathscr{F}\varphi \in \mathscr{S}(\mathfrak{n}^*)$ is defined by taking the usual Fourier transform of ψ:

$$\mathscr{F}\varphi(l) = \psi\hat{\ }(l) = \int_{\mathfrak{n}} \psi(X)\, e^{2\pi i\langle l, X\rangle}\, dX \qquad (\text{where } dX = dn \text{ modulo } exp). \tag{1}$$

Then Kirillov's formula says:

$$\operatorname{Tr}(\pi_\varphi) = \int_{\mathcal{O}} \psi\hat{\ }(l)\, d\mu_{\mathcal{O}}(l), \qquad \text{all} \quad \varphi \in \mathscr{S}(N), \tag{2}$$

where $\mu_{\mathcal{O}}$ is a suitably normalized version of the essentially unique N-invariant measure on $\mathcal{O}$. The canonical measure $\mu_{\mathcal{O}}$ does not depend on the choice of Haar measure dn used in defining π_φ, provided we use the corresponding Lebesgue measure dX on $\mathfrak{n}$ in defining $\psi\hat{\ } = \mathscr{F}\varphi$. The appropriate measure was first characterized in [10]; for a somewhat different description, see [2, Section 3].

One definition of a "Fourier transform" of a function φ on N uses the formula

$$F\varphi(\pi) = \operatorname{Tr}(\pi_\varphi), \qquad \pi \in N\hat{\ }. \tag{3}$$

This definition enables one to state the Fourier inversion formula for functions on N in a form like that for the abelian case:

$$\varphi(e) = \int_{N\hat{\ }} \operatorname{Tr}(\pi_\varphi)\, d\mu(\pi) \tag{4}$$

where μ is the so-called Plancherel measure on N. Dixmier [4] computes various examples and Kirillov [8] gave a general description of μ. A theorem due independently to Chevalley and Rosenlicht, to be discussed in Section 2, implies that if $d = 2k$ is the maximum dimension of orbits in $\mathfrak{n}^*$, then the orbits in an N-invariant Zariski open set $\mathscr{V} \subseteq \mathfrak{n}^*$ are all diffeomorphic to $\mathbf{R}^{2k}$ and may be parametrized by the points in a Zariski open subset $\mathscr{W} \subseteq \mathbf{R}^{n-2k}$. Kirillov showed that there is a rational function $q(\lambda)$ on $\mathscr{W}$ such that $d\mu(\pi_\lambda) = q(\lambda)\, d\lambda$, where $d\lambda$ = Lebesgue measure on $\mathbf{R}^{n-2k}$. In this picture, the canonical measure for an orbit $\mathcal{O}_\lambda \subseteq \mathscr{V}$ can be regarded as $1/q(\lambda)$ times Lebesgue measure on $\mathbf{R}^{2k}$, and (4) becomes a statement about direct integral decomposition of Lebesgue measure on $\mathbf{R}^n = \mathfrak{n}^*$, saying in effect that

$$\int_{\mathbf{R}^{n-2k}}^{\oplus} q(\lambda)\, \mu_{\mathcal{O}(\lambda)}\, d\lambda = \int_{\hat{N}}^{\oplus} \mu_{\mathcal{O}(\pi)}\, d\mu(\pi) = \text{Lebesgue measure on } \mathfrak{n}^*.$$

The representations not parametrized by orbits in $\mathscr{V}$ form a set of Plancherel measure zero. The above reasoning leads to a description of the space of functions $F\varphi(\lambda) = \mathrm{Tr}(\pi_\varphi{}^\lambda)$, $\lambda \in \mathscr{W}$, $\varphi \in \mathscr{S}(N)$. It is easy to check that $f: \mathscr{W} \to \mathbf{C}$ is of the form $f = F\varphi$ for some $\varphi \in \mathscr{S}(N) \Leftrightarrow q(\lambda)\, f(\lambda)$ extends to a Schwartz class function on $\mathbf{R}^{n-2k}$. Thus the function $q(\lambda)$ effectively determines the Fourier transforms of Schwartz functions when we take transforms as in (3).

We are interested in finding a comparable description of the full Fourier transform $\lambda \mapsto \pi_\varphi{}^\lambda$, $\lambda \in \mathscr{W}$. Any description of these operators is necessarily noncanonical since $\pi_\varphi{}^\lambda$ depends on the particular concrete model for the representation π^λ. As is well known, π^λ can be modeled in $L^2(\mathbf{R}^k)$, $k = \frac{1}{2}\dim(\mathcal{O}_\pi)$; however, even here there is considerable leeway in choosing the model. We shall choose the models so they vary smoothly in λ. Then, writing the operators $\pi_\varphi{}^\lambda$ as integral operators

$$[\pi_\varphi{}^\lambda f](u) = \int_{\mathbf{R}^k} K_\varphi(u, t; \lambda)\, f(t)\, dt, \qquad \text{all} \quad f \in L^2(\mathbf{R}^k), \quad \lambda \in \mathscr{W},$$

we shall regard $K_\varphi: \mathbf{R}^k \times \mathbf{R}^k \times \mathscr{W} \to \mathbf{C}$ as the Fourier transform of φ. The problem, then, is to describe $\{K_\varphi : \varphi \in \mathscr{S}(N)\}$. Notice that in this description we are concerned only with orbits in general position, those parametrized by $\mathscr{W}$.

Our approach is to find explicit formulas for the kernel functions $K_\varphi(u, t)$; when π^λ is allowed to vary over representations in general position, we obtain formulas for $K_\varphi(u, t, \lambda)$. At present, however, we are able to obtain a useful description for these transforms only in the special case when there is a single subalgebra $\mathfrak{m}$ that is (i) an ideal $\mathfrak{m} \lhd \mathfrak{n}$, (ii) maximal subordinate for all $l \in \mathfrak{n}^*$ in general position. [Then $\mathfrak{m}$ must actually be abelian; perhaps it is worth noting that such $\mathfrak{m}$ exists $\Leftrightarrow$ there exists even one l in general position having an abelian ideal as a maximal subordinate subalgebra.] These requirements are met in a number of interesting cases, such as the Heisenberg groups or the group of all upper triangular $n \times n$ matrices with 1's on the diagonal, and we have some indications that the main features of our theory can be carried over to general nilpotent Lie groups.

In Section 2 we compute the kernel functions $K_\varphi(u, t, \lambda)$, and then prove our main result characterizing the set of transforms as functions of $(u, t, \lambda) \in \mathbf{R}^k \times \mathbf{R}^k \times \mathscr{W} \cong \mathbf{R}^n$. The behavior of these "bare" kernel functions can be quite complicated, even in simple examples; we show that there is a single birational change of variables $A: \mathbf{R}^k \times \mathbf{R}^k \times \mathscr{W} \to \mathbf{R}^n$ such that the "rationalized" kernels $F\varphi = \tilde{K}_\varphi = K_\varphi \circ A^{-1}$ have an elegant characterization: we have $F(\mathscr{S}(N)) = \mathscr{S}(\mathbf{R}^n)$. The confusing behavior of the bare transforms K_φ is entirely due to the change of variable A. Moreover, the polynomial differential operators $\mathscr{P}(N)$ on $\mathscr{S}(N)$ carry over isomorphically to the polynomial operators $\mathscr{P}(\mathbf{R}^n)$ on $\mathscr{S}(\mathbf{R}^n)$ under the rationalized Fourier transform $F(\varphi) = \tilde{K}_\varphi$.

In Section 3 we compare our formulas for $\mathrm{Tr}(\pi_\varphi)$ with Kirillov's original trace formula. This leads to some curious identities for orbital integrals of Fourier transforms, which in some sense are nonlinear Poisson summation formulas.

We may note that the corresponding problem of characterizing Fourier transforms was solved for rank 1 semisimple Lie groups by Arthur [1]. There one may regard π_φ as an infinite matrix, using as basis vectors the vectors of various K-types (K being the maximal compact subgroup). No such basis is available for nilpotent groups, of course. Geller [4] has used a similar approach (fixing explicit bases in the representation spaces) to identify the Fourier transform of $\mathscr{S}(N)$ when N is the Heisenberg group.

2. A Fourier transform for $\mathscr{S}(N)$

We consider a fixed irreducible representation π of N and show that if the model for π is suitably chosen, there is a useful formula for the kernels of the trace class operators π_φ, $\varphi \in \mathscr{S}(N)$. Let π be associated with the orbit of $l \in \mathfrak{n}^*$ and let $\mathfrak{m}$ be a maximal subordinate subalgebra. Choose a weak Malcev basis $Y_1, \ldots, Y_m, X_1, \ldots, X_k$ for $\mathfrak{n}$ such that $\mathbf{R}$-span $\{Y_1, \ldots, Y_m\} = \mathfrak{m}$. (*Weak Malcev basis* means: each $\mathfrak{n}_j = \mathbf{R}$-span $\{X_1, \ldots, X_j\}$ is a subalgebra with $\mathfrak{n}_j$ an ideal in $\mathfrak{n}_{j+1}$ and $\mathfrak{n}_j \backslash \mathfrak{n}_{j+1} \cong \mathbf{R}$; a *strong Malcev basis* is one such that each $\mathfrak{n}_j$ is an ideal in $\mathfrak{n}$. For discussion, see [2], Section 3.) Using the basis we shall identify $\mathfrak{m}$ with $\mathbf{R}^m$ and, for convenience, will often write $l(s) = \langle l, \sum_i s_i Y_i \rangle$ for $l \in \mathfrak{m}^*$, $s \in \mathbf{R}^m$. Now we know that $M = \exp(\mathbf{R}Y_1) \cdots \exp(\mathbf{R}Y_m)$ and that $\Sigma = \exp(\mathbf{R}X_1) \cdots \exp(\mathbf{R}X_k)$ is a closed set cross-sectioning $M \backslash N$. Let π be regarded as induced from $\chi = (e^{2\pi i l}) \circ \log$ on M; π is modeled in a space $\mathscr{H}(\pi)$ of functions f on N that vary like χ along M-cosets, $f(mn) = \chi(m) f(n)$, with π acting on the right: $[\pi_x f](n) = f(nx)$. Define β: $\mathbf{R}^m \times \mathbf{R}^k \to N$,

$$\beta(s, t) = \beta(s_1, \ldots, s_m, t_1, \ldots, t_k) = \exp(s_1 Y_1) \cdots \exp(t_k X_k). \tag{6}$$

As in [2], the maps $s \mapsto \beta(s, 0)$ and $(s, t) \mapsto \beta(s, t)$ transfer Euclidean measures ds, $ds\, dt$ to Haar measures dm, dn on M, N, and dt to an N-invariant measure $d\dot{n}$ on $M \backslash N$. Using $d\dot{n}$ in the definition of norm, $\|f\|^2 = \int_{M\backslash N} |f(n)|^2\, d\dot{n}$, in $\mathscr{H}(\pi)$, the correspondence $f \mapsto \tilde{f}(t) = f(\beta(0, t))$ is an isometry from $\mathscr{H}(\pi)$ to $L^2(\mathbf{R}^k, dt)$, which we use to model π in $L^2(\mathbf{R}^k)$. For future reference, note that l kills all commutators of elements in $\mathfrak{m}$ so that $\langle l, \log \beta(s, 0) \rangle = \langle l, s_1 Y_1 * \cdots * s_m Y_m \rangle$ (Campbell–Hausdorff product) $= \langle l, \sum s_i Y_i \rangle = l(s)$ and

$$\begin{aligned} f(\beta(s, t)) &= f(\beta(s, 0)\, \beta(0, t)) = e^{2\pi i \langle l, \log \beta(s,0) \rangle} f(\beta(0, t)) \\ &= e^{2\pi i l(s)} f(\beta(0, t)) \end{aligned} \tag{7}$$

for all $f \in \mathscr{H}(\pi)$. For convenience, write $\gamma(t) = \beta(0, t)$ for $t \in \mathbf{R}^k$.

For $\varphi \in \mathscr{S}(N)$ use the Haar measure $dn = ds\, dt = dm\, d\dot{n}$ on N to define the operators $\pi_\varphi = \int_N \varphi(n)\, \pi_n\, dn$. For $\gamma(u) \in \Sigma = \beta(0, \mathbf{R}^k)$ and bounded continuous $f \in \mathscr{H}(\pi)$, we have an absolutely convergent integral

$$[\pi_\varphi f](\gamma(u)) = \int_N f(\gamma(u)n)\, \varphi(n)\, dn$$
$$= \int_N f(n)\, \varphi(\gamma(u)^{-1}n)\, dn.$$

Now use the unique splitting $n = m \cdot \gamma(t)$, which provides a polynomial diffeomorphism $N \approx M \times \mathbf{R}^k$, and Fubini to write this integral as

$$\int_M \int_{\mathbf{R}^k} f(m\gamma(t))\, \varphi(\gamma(u)^{-1}m\gamma(t))\, dm\, dt$$
$$= \int_{\mathbf{R}^k} f(\gamma(t)) \left[\int_M \chi(m)\, \varphi(\gamma(u)^{-1}m\gamma(t))\, dm\right] dt$$

where dm is chosen as stated in the theorem ($dn = dm\, d\dot{n}$). Obviously, if we parametrize M via β: $\mathbf{R}^m \times (0) \to M$, we get $\chi(m) = e^{2\pi i \langle l, \log\beta(s,0)\rangle} = e^{2\pi i l(s)}$ and $m\gamma(t) = \beta(s, 0)\, \beta(0, t) = \beta(s, t)$. We may write this in terms of partial Fourier transforms. Let $G_u(s, t) = \gamma(u)^{-1}\, \beta(s, t) = \gamma(u)^{-1}\, \beta(s, 0)\, \gamma(t)$, and let $\mathscr{F}_1\psi$ be the partial Fourier transform in the first m variables regarding $\mathfrak{m}^* = (\mathbf{R}^m)^*$ via the pairing $\langle l, s\rangle = l(s) = \langle l, \sum s_i Y_i\rangle$; then

$$\mathscr{F}_1\psi(l', t) = \int_{\mathbf{R}^m} \psi(s, t)\, e^{2\pi i \langle l', s\rangle}\, ds, \qquad \text{all} \quad \psi \in \mathscr{S}(\mathbf{R}^m \times \mathbf{R}^k), \quad l' \in \mathfrak{m}^*$$

and we get

$$\begin{aligned}[\pi_\varphi f](\gamma(u)) &= \int_{\mathbf{R}^k} \int_{\mathbf{R}^m} \varphi(\gamma(u)^{-1}\beta(s, t))\, e^{2\pi i l(s)} f(\gamma(t))\, ds\, dt \\ &= \int_{\mathbf{R}^k} \mathscr{F}_1(\varphi \circ G_u)(l \mid \mathfrak{m}; t)\, f(\gamma(t))\, dt\end{aligned} \tag{8}$$

Notice that

$$K_\varphi(u, t) = \int_M \chi(m)\, \varphi(\gamma(u)^{-1}m\gamma(t))\, dm = \mathscr{F}_1(\varphi \circ G_u)(l \mid \mathfrak{m}; t) \tag{9}$$

is continuous on $\mathbf{R}^{2k}$. Since π_φ is known to be trace class on $L^2(\mathbf{R}^k)$, K_φ must be its kernel (and be square integrable on $\mathbf{R}^{2k}$, among other things).

Formula (9) is not too helpful in answering questions. For example, it is not so clear that K_φ is Schwartz class on $\mathbf{R}^{2k}$, a fact which Howe [7] discuss by other means. For our purposes here, keeping track of how the kernels vary as the representation changes, it seems equally intractable without additional hypotheses. If M is normal, the situation is clearer. Starting from (8), define polynomial maps ξ, η: $\mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^m$ (resp. $\mathbf{R}^k$) such that $\gamma(u)^{-1}\, \gamma(t) = \beta(\xi, 0) \times$

$\beta(0, \eta) = \beta(\xi, \eta)$. Write $l \cdot n = \mathrm{Ad}^*(n^{-1})l$ (a right action of N on $\mathfrak{n}^*$). Then we get

$$\begin{aligned}\mathscr{F}_1(\varphi \circ G_u)(l \mid \mathfrak{m}, t) &= \int_{\mathbf{R}^m} \varphi(\gamma(u)^{-1}\beta(s, 0)\, \beta(0, t))\, e^{2\pi i\langle l, \log\beta(s,0)\rangle}\, ds \\ &= \int \varphi(\beta(s, 0)\, \gamma(u)^{-1}\gamma(t))\, e^{2\pi i\langle l, \log\gamma(u)\beta(s,0)\gamma(u)^{-1}\rangle}\, ds \\ &= \int \varphi(\beta(s, 0)\, \beta(\xi, \eta))\, e^{2\pi i\langle l, \mathrm{Ad}(\gamma(u))\log\beta(s,0)\rangle}\, ds \\ &= \int \varphi(\beta(s, 0)\, \beta(0, \eta))\, e^{2\pi i\langle l\cdot\gamma(u), \log\beta(s,0)\beta(\xi,0)^{-1}\rangle}\, ds \\ &= e^{-2\pi i\langle l\cdot\gamma(u), \log\beta(\xi,0)\rangle} \int \varphi \circ \beta(s, \eta)\, e^{2\pi i\langle l\cdot\gamma(u), \log\beta(s,0)\rangle}\, ds \\ &= e^{-2\pi i\langle l\cdot\gamma(u), \xi(u,t)\rangle} \mathscr{F}_1(\varphi \circ \beta)(l \cdot \gamma(u) \mid \mathfrak{m}, \eta(u, t)).\end{aligned}$$

Now the function to which $\mathscr{F}_1$ is applied no longer varies with u, and the kernel function is

$$K_\varphi(u, t) = e^{-2\pi i\langle l\cdot\gamma(u), \xi(u,t)\rangle} \mathscr{F}_1(\varphi \circ \beta)(l \cdot \gamma(u) \mid \mathfrak{m}, \eta(u, t)). \tag{10}$$

Using formulas (8) and (10), we will show that some interesting things can be said about the joint behavior of the kerenels in u, t, and l.

We digress for a moment to recall the Rosenlicht–Chevalley Theorem, as given in [12, pp. 55–58]. Let N act unipotently on a real vector space V and let $e_1, \ldots, e_n$ be a Jordan–Hölder basis, so that the subspaces $V_i = \mathbf{R}$-span $\{e_i, \ldots, e_n\}$ are N-invariant. For convenience, write $x = \sum x_j e_j$ as $(x_1, \ldots, x_n)$. Suppose that the maximum dimension of any N-orbit in V is d. The theorem says that there is an N-invariant Zariski open set $\mathscr{V} \subseteq V$ and a set of n functions $f_1, \ldots, f_n$ in $n + d$ variables $(x, t) = (x_1, \ldots, x_n, t_1, \ldots, t_d)$ such that:

(a) Each function is rational in x and polynomial in t:

$$f_j(x, t) = \sum \{c_\alpha(x)\, t^\alpha : \alpha \in \mathbf{Z}_+{}^d\}$$

with $c_\alpha(x)$ rational on $\mathbf{R}^n$.

(b) For fixed $x \in \mathscr{V}$, $x \cdot N = \{(f_1(x, t), \ldots, f_n(x, t)) : t \in \mathbf{R}^d\}$.

(c) Each f_j is nonsingular on the fiber $\{x\} \times \mathbf{R}^d$ if $x \in \mathscr{V}$. (Note : the singular set is saturated in the t-direction.)

(d) There are indices $j_1 < j_2 < \cdots < j_d$ such that $f_{j_i}(x, t) = t_i$, and for each fixed x, $f_j(x, t)$ depends only on the t_i such that $j_i \geqslant j$. For fixed t, and $j \neq j_k$ $(1 \leqslant k \leqslant d)$, $f_j(x, t) = x_j \pmod{x_1, \ldots, x_{j-1}}$.

Let $W \subseteq V$ be the subspace of codimension d defined by $x_{j_i} = 0$, $1 \leqslant i \leqslant d$. It is clear that each N-orbit in $\mathscr{V}$ meets W in exactly one point. Let $\mathscr{W} =$

$\mathscr{V} \cap W$. By renaming variables (and functions), we find that $F = (f_1, ..., f_n)$: $\mathscr{W} \times \mathbf{R}^d \to \mathscr{V}$ is a birational isomorphism, nonsingular in each direction, rational in the $\mathscr{W}$-variables $(x_1, ..., x_{n-d})$ and polynomial in $(t_1, ..., t_d)$. Moreover, if N acts freely on generic orbits, there is a birational isomorphism Φ of N with $\mathbf{R}^d$ such that if $\Phi(n) = t$, and $x \cdot n = F(x, \Phi(n))$ for all $x \in \mathscr{W}$, $n \in N$.

THEOREM 2.1. *Suppose there exists a single ideal $\mathfrak{m} \lhd \mathfrak{n}$ that is maximal subordinate for all functionals in general position, $l \in \mathfrak{n}^*_{gp}$. Let $Y_1, ..., Y_m, X_1, ..., X_k$ be a strong Malcev basis passing through $\mathfrak{m}$ and use these coordinates to model all representations π^l in $L^2(\mathbf{R}^k)$, as above. For a suitable choice of* $\mathrm{Ad}^*(N)$*-invariant Zariski open set $\mathscr{U}$ defining "general position", there is a subspace $W \subseteq \mathfrak{n}^*$ such that* (i) *$\mathscr{W} = W \cap \mathscr{U}$ meets each orbit in $\mathscr{U}$ precisely once, and* (ii) *the resulting kernel functions $K_\varphi(u, t, l)$ defined on $\mathbf{R}^k \times \mathbf{R}^k \times \mathscr{W}$ have the form*

$$K_\varphi(u, t, l) = e^{-2\pi i\langle l\cdot\gamma(u), \xi(u,t)\rangle}[\mathscr{F}_1(\varphi \circ \beta)]\, A(u, t, l) \tag{10}$$

where $\xi: \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^m$ is a polynomial, $\mathscr{F}_1$ is the partial Fourier transform

$$\mathscr{F}_1(\psi)(l', t) = \int_{\mathbf{R}^k} \psi(s, t)\, e^{2\pi i\langle l', \Sigma s_j Y_j\rangle}\, ds$$

for all $\psi \in \mathscr{S}(\mathbf{R}^k \times \mathbf{R}^k)$, $l' \in \mathfrak{m}^*$, and where $A: \mathbf{R}^k \times \mathbf{R}^k \times \mathscr{W} \to \mathfrak{m}^* \times \mathbf{R}^k$ is a polynomial map that is a birational isomorphism between these two concrete versions of $\mathbf{R}^n$. For all $\varphi \in \mathscr{S}(N)$, $\tilde{K}_\varphi(l', t) = \mathscr{F}_1(\varphi \circ \beta)$ is a Schwartz function on $\mathfrak{m}^* \times \mathbf{R}^k$; conversely, a function f on $\mathbf{R}^k \times \mathbf{R}^k \times \mathscr{W}$ is a Fourier transform, $f = K_\varphi$ for some $\varphi \in \mathscr{S}(N)$, if and only if there is a Schwartz function

$$\tilde{f} \in \mathscr{S}(\mathfrak{m}^* \times \mathbf{R}^k)$$

such that $f = e^{-2\pi i\langle l\cdot\gamma(t), \xi\rangle}[\tilde{f} \circ A](u, t, l)$.

Proof. Let $l_1, ..., l_m, ..., l_{m+k}$ be the dual basis to $Y_1, ..., Y_m, ..., X_k$; this is a Jordan–Holder basis for the action of N on $\mathfrak{n}^*$. Let us identify $\mathfrak{m}^*$ with $\mathbf{R}$-span$\{l_1, ..., l_m\}$; then $l_1, ..., l_m$ is a Jordan–Holder basis for the induced action $M\backslash N \times \mathfrak{m}^* \to \mathfrak{m}^*$. Since $\mathfrak{m}$ is an ideal, the natural projection $p: \mathfrak{n}^* \to \mathfrak{m}^*$, $p(l) = l \mid m$, maps N-orbits to $M\backslash N$-orbits and is equivariant in an obvious sense. Orbits $\mathscr{O} \subseteq \mathfrak{n}^*$ in general position are saturated by the N-invariant subspace $\mathfrak{m}^\perp = \mathbf{R}$-span$\{l_{m+1}, ..., l_{m+k}\}$, in that $l \in \mathscr{O} \Rightarrow l + \mathfrak{m}^\perp \subseteq \mathscr{O}$, so p maps distinct N-orbits in general position to distinct $M\backslash N$-orbits.

Suppose $\mathscr{U}_1$ is an $M\backslash N$-invariant Zariski open set in $\mathfrak{m}^*$, and $W_1 \subseteq \mathfrak{m}^*$ a subspace which meets each of the orbits in $\mathscr{U}_1$ in just one point, as in Chevalley–Rosenlicht; then $\mathscr{W}_1 = W_1 \cap \mathscr{U}_1$ is a Zariski open set in W_1 which cross-sections the orbits in $\mathscr{U}_1$. Now $\mathscr{U} = p^{-1}(\mathscr{U}_1)$ is an N-invariant Zariski open set in $\mathfrak{n}^*$ and, by suitably altering $\mathscr{U}_1$, we can insure that $\mathscr{U}$ consists entirely of $\mathfrak{m}^\perp$-

saturated orbits of maximal (generic) dimension. [In fact, since $\mathfrak{m}$ is maximal isotropic generically, it is isotropic (if not maximal) for $B_l(X, Y) = \langle l, [X, Y]\rangle$ for all $l \in \mathfrak{n}^*$. It is not hard to see that $\mathfrak{m}$ fails to be maximal isotropic for $l \Leftrightarrow$ the matrix $M(l) = \{B_l(X_i, Y_j) : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant m\}$ has rank $<k$. The l for which this happens form a Zariski closed set with nonempty complement $\mathscr{U}_0$. Evidently, $\mathscr{U}_0$ is N-invariant and consists of $\mathfrak{m}^\perp$-saturated orbits, and $p(\mathscr{U}_0)$ is an $M\backslash N$-invariant Zariski open set in $\mathfrak{m}^*$. Replace $\mathscr{U}_1$ with $\mathscr{U}_1 \cap p(\mathscr{U}_0)$ if necessary, and $\mathscr{U}$ by $\mathscr{U} \cap \mathscr{U}_0$; W_1 still serves as a cross-section to the orbits in $\mathscr{U}_1$.] Now $W = p^{-1}(W_1) \cap \mathbf{R}\text{-span}\{l_1, ..., l_m\}$ is a subspace which cross-sections the orbits in $\mathscr{U}$; note that p: $W \to W_1$ is a bijective linear map.

In this situation, formula (10) holds simultaneously for all $l \in \mathscr{W}$. To get (11) we take $\tilde{K}_\varphi = \mathscr{F}_1(\varphi \circ \beta)$ and $A(u, t, l) = (l \cdot \gamma(t) \mid \mathfrak{m}, \eta(u, t)) = (p(l) \cdot \gamma(t), \eta(u, t))$ where ξ, η are the polynomial maps defined so that $\beta(0, u)^{-1}\, \beta(0, t) = \beta(\xi, \eta)$; by slight abuse of notation we write $l' \cdot n = l' \cdot Mn$ for the action of $M\backslash N$ on $l' \in \mathfrak{m}^*$. Clearly the map $\varphi \mapsto \tilde{K}_\varphi$ is a topological isomorphism of Schwartz spaces, so it remains only to show that A is a birational isomorphism. Since p: $W \to W_1$ is a linear isomorphism, we may regard A as a map A: $\mathbf{R}^k \times \mathbf{R}^k \times W_1 \to \mathfrak{m}^* \times \mathbf{R}^k$.

For any Malcev basis of $\mathfrak{n}$, such as $Y_1, ..., X_k$, the associated map β: $\mathbf{R}^{m+k} \to N$ is polynomial with polynomial inverse (if we identify N with $\mathfrak{n}$ via the exponential map). Modulo M we get a related map $\tilde{\beta}$: $\mathbf{R}^k \to M\backslash N$ such that $\tilde{\beta}(t) = M \cdot \beta(0, t)$; $\tilde{\beta}$ is also a polynomial map with polynomial inverse.

We claim that the transformation $(u, t) \to (\eta, t)$ is a polynomial from $\mathbf{R}^{2k}$ to itself, with polynomial inverse. Clearly the forward map is polynomial, and given (η, t) we may solve for $\beta(0, u)$ as follows:

$$\beta(0, u) = \beta(0, \eta)^{-1}\, \beta(\xi, 0)\, \beta(0, t).$$

Modulo M, we get $\tilde{\beta}(u) = \tilde{\beta}(\eta)^{-1}\, \tilde{\beta}(t)$, so $u = \tilde{\beta}^{-1}(\tilde{\beta}(u)^{-1}\, \tilde{\beta}(t))$, a polynomial.

Next, the map $(t, l) \to l' = l \cdot \gamma(t)$ of $\mathbf{R}^k \times W_1 \to \mathfrak{m}^*$ is birational (the forward map being polynomial) by the Chevalley–Rosenlicht theorem, since $M\backslash N$ acts freely on all orbits in $\mathscr{U}_1$ [by dimension counting and the fact that $l \cdot M = l + \mathfrak{m}^\perp$ if $\mathfrak{m}$ is maximal subordinate for l, see [12]]. We may now solve $A(u, t, l) = (l', \eta)$ to find (t, l) as a rational function of l', and then (u, t) as a polynomial function of (t, η). Q.E.D.

Notice that the ideal $\mathfrak{m}$ must actually be abelian: it is subordinate for *all* $l \in \mathfrak{n}^*$, so $\langle l, [\mathfrak{m}, \mathfrak{m}]\rangle = 0$ for all l and $[\mathfrak{m}, \mathfrak{m}] = 0$. Thus the action of M on $\mathfrak{m}^*$ is trivial, yielding the quotient action $M\backslash N \times \mathfrak{m}^* \to \mathfrak{m}^*$, as above. The function $K_\varphi(u, t, l)$ is the natural candidate for a Fourier transform of $\varphi \in \mathscr{S}(N)$, but it has some unpleasant properties. Even if N is the Heisenberg group (see Example 2.3 below), K_φ is not generally a Schwartz function in the full set of variables (u, t, l), though it is Schwartz in (u, t) for fixed $l \in \mathfrak{w}$. From (11) it is evident that the singular behavior or of K_φ is essentially due to the map A. On the other hand the "rationalized Fourier transform" $\tilde{K}_\varphi = \mathscr{F}_1(\varphi \circ \beta) = K_\varphi \circ A^{-1}$, which differs from K_φ by a birational change of variables, is a much more pleasant object for which we have a simple analog of the Paley–Wiener theorem. Various refinements are possible: for example, if $\varphi \in C_c^k(N)$ with $k \geqslant n$, then π_φ is trace class, formula (11) is still valid, and

$$K_\varphi = e^{2\pi i\langle l\cdot\gamma(u),\xi\rangle}[\tilde{K}_\varphi \circ A].$$

The degree of differentiability of φ may be determined by examining the behavior of $\tilde{K}_\varphi$, using the tools of classical Fourier analysis.

If we write $F\varphi = \tilde{K}_\varphi(u, t, l)$ for the rationalized transform, we may ask how differential operators with polynomial coefficients $T \in \mathscr{P}(N)$ transform to operators $\tilde{T}$ such that $\tilde{T}(F\varphi) = F(T\varphi)$, all $\varphi \in \mathscr{S}(N)$.

COROLLARY 2.2. *In the setting of Theorem* 2.1, $\mathscr{P}(N)$ *transforms isomorphically to the set of polynomial operators* $\mathscr{P}(\mathfrak{m}^* \times \mathbf{R}^k)$.

Proof. First consider the transformation $T \to T'$ such that $T'(\varphi \circ \beta) = T(\varphi) \circ \beta$. It is more convenient to deal with the related map $\beta' = \log \circ \beta \circ j^{-1}$: $\mathfrak{n} \to \mathfrak{n}$ (where $j(s, t) = \sum s_i Y_i + \sum t_j X_j$) and show that $\mathscr{P}(\mathfrak{n})$ maps to $\mathscr{P}(\mathfrak{n})$. Since β' and its inverse are polynomial maps, so are their Jacobian determinants, which must then be constant. The constant is 1 (we can actually compute it at the origin in $\mathfrak{n}$). Write $\{W_1, \ldots, W_n\} = \{Y_1, \ldots, X_k\}$ for the basis. The coordinate multiplications $M_i\psi = w_i\psi$ clearly map to operators of the same kind: $M_i'\psi(W) = p_i(W)\,\psi(W)$, where $\beta'(\sum t_i W_i) = t_1 W_1 * \cdots * t_n W_n = \sum p_j(t)\, W_j$; since β' is invertible the entire subalgebra of multiplication operators $\mathcal{O}\!\mathcal{l} \subseteq \mathscr{P}$ is preserved. As for the differentiations $D_i = \partial/\partial t_i$ with respect to the additive coordinates $W = \sum t_i W_i$, we have $D_k(\psi \circ \beta') = \sum_{i=1}^n [(D_i\psi) \circ \beta'][\partial p_i/\partial t_k]$. Solving for $D_i'(\psi \circ \beta') = (D_i\psi) \circ \beta'$, we see that each D_i' is a sum with polynomial coefficients of $D_1, \ldots, D_n$, and conversely. Thus $T \to T'$ is an isomorphism from $\mathscr{P}(N)$ to $\mathscr{P}(\mathbf{R}^m \times \mathbf{R}^k)$. Next, the partial Fourier transform $\mathscr{F}_1$ induces an isomorphism $T' \to T''$ from $\mathscr{P}(\mathbf{R}^m \times \mathbf{R}^k)$ to $\mathscr{P}(\mathfrak{m}^* \times \mathbf{R}^k)$; this amounts to a straightforward question about such transforms in $\mathscr{P}(\mathbf{R}^n)$. Finally, $F(T\varphi) = \tilde{K}_{T\varphi} = \mathscr{F}_1(T\varphi \circ \beta) = \mathscr{F}_1(T'(\varphi \circ \beta)) = T''\mathscr{F}_1(\varphi \circ \beta) = \tilde{T}(\tilde{K}_\varphi) = \tilde{T}(F\varphi)$ so $T \mapsto \tilde{T} = T''$ is an isomorphism. Q.E.D.

As the following example shows, it is possible to define operators $\hat{T}$ on the actual kernels such that $\hat{T}(K_\varphi) = K_{T\varphi}$, but the homomorphism $T \mapsto \hat{T}$ yields fairly messy operators in the variables (u, t, l). This difficulty disappears when we pass over to the corresponding operators $\tilde{T}$ on the rationalized kernels $\tilde{K}_\varphi$.

EXAMPLE 2.3. Let $\mathfrak{n}$ be the three dimensional Heisenberg algebra with basis $\{Z, Y, X\}$ such that $[X, Y] = Z$, and let $\mathfrak{m} = \mathbf{R}\text{-span}\{Z, Y\}$. For $W \subseteq \mathfrak{n}^*$

we may take $\mathbf{R}Z^*$ (dual basis in $\mathfrak{n}^*$) and $\mathscr{W} = \mathbf{R}Z^* \sim (0)$. Write $\psi(z, y, x)$ for $\psi(zZ + yY + xX)$, $\psi \in \mathscr{S}(\mathfrak{n})$. We compute the Fourier transforms of $\varphi = \psi \circ \log$ in terms of ψ. Since $\beta(z, y, x) = \exp((z - \frac{1}{2}xy)Z + yY + xX)$ we get $\xi(u, t) \equiv 0$ (as is always the case if $\mathfrak{n}$ is a semidirect product $\mathfrak{n} = \mathfrak{m} + \mathfrak{h}$ and the basis is such that $\mathbf{R}\text{-span}\{X_1, \ldots, X_k\} = \mathfrak{h}$) and $\eta(u, t) = t - u$. Moreover, $(\sigma Z^* + \tau Y^*) \cdot \gamma(u) = \mathrm{Ad}^*(\gamma(u)^{-1})(\sigma Z^* + \tau Y^*) = \sigma Z^* + (u\sigma + \tau) Y^*$; taking orbit representatives $l = \lambda Z^* \in \mathscr{W} (\lambda \neq 0)$, we get $l \cdot \gamma(u) = \lambda Z^* + u\lambda Y^*$. Now $\varphi \circ \beta(z, y, x) = \psi \circ \log \circ \beta = \psi(z - \frac{1}{2}xy, y, x)$, so that the rationalized kernels have the form

$$\tilde{K}_\varphi(\sigma Z^* + \tau Y^* \mid \mathfrak{m}, x) = \int\int \psi\left(z - \frac{xy}{2}, y, x\right) e^{2\pi i(\sigma z + \tau y)}\, dz\, dy$$

$$= \mathscr{F}_1\psi\left(\sigma, \tau + \frac{\sigma x}{2}, x\right), \qquad \text{all} \quad \varphi \in \mathscr{S}(\mathfrak{n}) = \mathscr{S}(\mathbf{R}^3)$$

where $\mathscr{F}_1\psi$ is the Euclidean Fourier transform in the first two variables. In the present situation, $A(u, t, \lambda Z^*) = (\lambda Z^* \cdot \gamma(u) \mid \mathfrak{m}, t - u) = (\lambda Z^* + \lambda u Y^* \mid \mathfrak{m}, t - u)$ and $A^{-1}(\sigma Z^* + \tau Y^* \mid \mathfrak{m}, x) = (\tau/\sigma, x + (\tau/\sigma), \sigma)$, so the actual kernels have the form

$$K_\varphi(u, t, \lambda Z^*) = \mathscr{F}_1\psi\left(\lambda, \frac{\lambda}{2}(t + u), t - u\right), \qquad \text{all} \quad \psi \in \mathscr{S}(\mathfrak{n}).$$

Still identifying N, $\mathscr{S}(N)$ with $\mathfrak{n}$, $\mathscr{S}(\mathfrak{n})$ via the exponential map, we next compute successive transforms of the basic polynomial operators $M_1\psi = z\psi, \ldots, M_3\psi = x\psi$, $D_1\psi = \partial\psi/\partial z, \ldots, D_3\psi = \partial\psi/\partial x$. We want $T'(\psi \circ \beta) = (T\psi) \circ \beta$; thus,

$$M_1'\psi = \left(z - \frac{xy}{2}\right)\psi(z, y, x) \quad M_2'\psi = y\psi(z, y, x) \qquad M_3'\psi = x\psi(z, y, x)$$

$$D_1'\psi = \frac{\partial\psi}{\partial z} \qquad D_2'\psi = \left(\frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}\right)\psi \quad D_3'\psi = \left(\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}\right)\psi.$$

In $\mathfrak{m}^* \times \mathbf{R}$ we use coordinates (σ, τ, r), taking $\sigma Z^* + \tau Y^* \mid \mathfrak{m}$ as a typical element of $\mathfrak{m}^*$. If $\psi \in \mathscr{S}(\mathfrak{m}^* \times \mathbf{R})$, we want $\tilde{T}(\mathscr{F}_1(\psi \circ \beta)) = \mathscr{F}_1(T'(\psi \circ \beta)) = \mathscr{F}_1((T\psi) \circ \beta)$; thus,

$$\tilde{M}_1\psi = \frac{1}{2\pi i}\left(\frac{\partial}{\partial\sigma} - \frac{r}{2}\frac{\partial}{\partial\tau}\right)\psi \quad \tilde{M}_2\psi = \frac{1}{2\pi i} \cdot \frac{\partial\psi}{\partial\tau} \qquad M_3\psi = r\psi$$

$$\tilde{D}_1\psi = 2\pi i\sigma\psi \qquad \tilde{D}_2\psi = 2\pi i\left(t - \frac{\sigma r}{2}\right)\psi \quad \tilde{D}_3\psi = \left(\frac{\partial}{\partial r} - \frac{\sigma}{2}\frac{\partial}{\partial\tau}\right)\psi.$$

If these operators are regarded as acting on the actual kernels $K = K_\varphi(u, t, \lambda Z^*)$, we have $\hat{T}K_\psi = K_{T\psi} = (\tilde{T}(K_\psi)) \circ A = (\tilde{T}(\tilde{K}_\psi \circ A^{-1})) \circ A$, or

$$\hat{M}_1 K = \frac{1}{2\pi i}\left[\frac{\partial K}{\partial \lambda} - \left(\frac{u+t}{2\lambda}\right)\left(\frac{\partial K}{\partial u} - \frac{\partial K}{\partial t}\right)\right]$$

$$\hat{M}_2 K = \frac{1}{2\pi i}\frac{1}{\lambda}\left(\frac{\partial K}{\partial u} + \frac{\partial K}{\partial t}\right) \qquad \hat{M}_3 K = (t-u)K$$

$$\hat{D}_1 K = 2\pi i \lambda K \qquad \hat{D}_2 K = 2\pi i \frac{\lambda}{2}(3u - t)K \qquad \hat{D}_3 K = \frac{1}{2}\left(\frac{\partial K}{\partial t} - \frac{\partial K}{\partial u}\right).$$

3. Orbital Integral Formulas for Kernels

We retain the notation of Section 2. We shall examine the kernels $K_\varphi(u, t)$ associated with a single representation π induced from a normal maximal subordinate subgroup M, and show that K_φ may be expressed in terms of integrals related to the orbit $\mathcal{O} \subseteq \mathfrak{n}^*$ corresponding to π. Along the way we get some interesting identities concerning the Fourier transform $\mathscr{F}$: $\mathscr{S}(N) \to \mathscr{S}(\mathfrak{n}^*)$, $\mathscr{F}(\varphi) = (\varphi \circ \exp)\hat{} = \int_\mathfrak{n} e^{2\pi i \langle l, W\rangle} \varphi \circ \exp(W)\, dW$.

To start, note that K_φ can be written in coordinate free form as a function on $N \times N$ varying like $\chi \times \bar{\chi}$ along $M \times M$-cosets, from which we compute π_φ as

$$[\pi_\varphi f](n_1) = \int_{M\backslash N} K_\varphi(n_1, n_2) f(n_2)\, d\dot{n}, \qquad \text{all continuous} \quad f \in \mathscr{H}(\pi).$$

We must specify dn on N and invariant measure $d\dot{n}$ on $M\backslash N$ in advance. These determine a measure dm on M such that $dn = dm\, d\dot{n}$; in the following discussion we take dW on $\mathfrak{n}$ so this Lebesgue measure transforms to dn under the exponential map. Given any cross section Σ to $M\backslash N$, K_φ is determined by its values on $\Sigma \times \Sigma$.

THEOREM 3.1. *Let $l \in \mathfrak{n}^*$ have a maximal subordinate subalgebra $\mathfrak{m}$ that is an ideal in $\mathfrak{n}$. Let invariant measures dn, $d\dot{n}$ be given. Then there is a polynomial map Q: $N \times N \to \mathfrak{n}$ and a polynomial diffeomorphism δ: $N \to N$ such that*

$$K_\varphi(n_1, n_2) = \int_{R_l\backslash M} \mathscr{F}(\varphi \circ \delta)(l \cdot m \cdot n_2)\, e^{2\pi i \langle l \cdot m \cdot n_2, Q(n_1, n_2)\rangle}\, d\dot{m}, \text{ all } \varphi \in \mathscr{S}(N). \quad (12)$$

Here $R(l) = \exp(\mathfrak{r}(l))$ where $\mathfrak{r}(l) =$ radical of l, and $d\dot{m}$ is chosen so that $d\dot{m}\, d\dot{n}$ on $R(l)\backslash N$ corresponds to the canonical measure $d\mu_\mathcal{O}$ on $\mathcal{O} = \mathrm{Ad}^*(N)l$ under $R(l)n \mapsto \mathrm{Ad}^*(n^{-1})l = l \cdot n$. The maps Q and δ do not depend on l, so (12) is valid simultaneously for all $l \in \mathfrak{n}^*$ for which $\mathfrak{m}$ is maximal subordinate.

Proof. Take a weak Malcev basis $\{Y_1', \ldots, Y_r', Y_1, \ldots, Y_k, X_1, \ldots, X_k\}$ passing through $\mathfrak{r}(l) = \mathbf{R}\text{-span}\{Y_1, \ldots, Y_r\}$ and then $\mathfrak{m}$. Using this basis we may transfer the Lebesgue measures ds, dt, and $ds\, dt$ on $\mathbf{R}^m$, $\mathbf{R}^k$, $\mathbf{R}^m \times \mathbf{R}^k$ to invariant measures dm, $d\dot{n}$, dn on M, $M\backslash N$, N in two ways, depending on whether we impose additive or Malcev coordinates in N; the outcome is the same either way. Assume the basis adjusted (by scaling) so dn, $d\dot{n}$ are the Haar measures given in the Theorem.

Define $j\colon \mathbf{R}^m \times \mathbf{R}^k \to \mathfrak{n}$ so $j(s, t) = \sum s_i Y_i + \sum t_j X_j$, and $J\colon N \to N$ so $J(mx) = xm$ for $m \in M$, $x \in \Sigma = \beta(0, \mathbf{R}^k)$. Then let $\delta = J \circ \beta \circ j^{-1} \circ \log\colon N \to N$, $\delta' = J \circ \beta \circ j^{-1}\colon \mathfrak{n} \to N$. There are polynomial maps $\sigma, \tau\colon \mathbf{R}^m \times \mathbf{R}^k \to \mathbf{R}^m$ (resp. $\mathbf{R}^k$) such that $\gamma(u)^{-1}\gamma(t) = \beta(0, \tau)\,\beta(\sigma, 0) = J(\beta(\sigma, \tau))$. Modeling π in $L^2(\mathbf{R}^k)$ via β, we start with formula (8) to get

$$
\begin{aligned}
K_\varphi(u, t) &= \int_{\mathbf{R}^m} \varphi(\gamma(u)^{-1}\beta(s, 0)\,\gamma(t))\, e^{2\pi i\langle l, \log\beta(s,0)\rangle}\, ds \\
&= \int \varphi(\gamma(u)^{-1}\gamma(t)\,\beta(s, 0))\, e^{2\pi i\langle l, \log\gamma(t)\beta(s,0)\gamma(t)^{-1}\rangle}\, ds \\
&= \int \varphi(\beta(0, \tau)\,\beta(\sigma, 0)\,\beta(s, 0))\, e^{2\pi i\langle l\cdot\gamma(t), \log\beta(s,0)\rangle}\, ds \\
&= \int \varphi(\beta(0, \tau)\,\beta(s, 0))\, e^{2\pi i\langle l\cdot\gamma(t), -\log\beta(\sigma,0)*\log\beta(s,0)\rangle}\, ds \\
&= e^{-2\pi i\langle l\cdot\gamma(t), \sigma\rangle} \int \varphi \circ J(\beta(s, \tau))\, e^{2\pi i\langle l\cdot\gamma(t), \log\beta(s,0)\rangle}\, ds.
\end{aligned}
$$

For clarity, let us write $M(s) = j(s, 0)$, $X(t) = j(0, t)$; let $\mathfrak{x} = \mathbf{R}\text{-span}\{X_1, \ldots, X_k\}$. Since $l \mid \mathfrak{m}$ is a character, $\langle l \cdot \gamma(t), \log\beta(s, 0)\rangle = \langle l \cdot \gamma(t), M(s)\rangle$. Now $\beta(s, \tau) = \beta \circ j^{-1}(M(s) + X(\tau))$, so the integral may be written as

$$
\begin{aligned}
&e^{-2\pi i\langle l\cdot\gamma(t), \sigma\rangle} \int_{\mathbf{R}^m} \varphi \circ J \circ \beta \circ j^{-1}(M(s) + X(\tau))\, e^{2\pi i\langle l\cdot\gamma(t), M(s)\rangle}\, ds \\
&\quad = e^{-2\pi i\langle l\cdot\gamma(t), \sigma\rangle} \int_{\mathfrak{m}} \varphi \circ \delta'(M + X(\tau))\, e^{2\pi i\langle l\cdot\gamma(t), m\rangle}\, dM \qquad (\text{where } dM = ds) \\
&\quad = e^{-2\pi i\langle l\cdot\gamma(t), \sigma\rangle} \mathscr{F}_1(\varphi \circ \delta')(l \cdot \gamma(t) \mid \mathfrak{m}, X(\tau)) \\
&\quad = e^{-2\pi i\langle l\cdot\gamma(t), \sigma\rangle} \int_{\mathfrak{m}^\perp} \mathscr{F}_2\mathscr{F}_1(\varphi \circ \delta')(l \cdot \gamma(t) \mid \mathfrak{m}, l')\, e^{-2\pi i\langle l', X(\tau)\rangle}\, dl'
\end{aligned}
$$

where dl' is the measure on $\mathfrak{m}^\perp = (\mathfrak{m}\backslash\mathfrak{n})^* = \mathfrak{x}^*$ dual under the Fourier transform to the Euclidean measure dX on $\mathfrak{x}$ corresponding to the basis $\{X_1, \ldots, X_k\}$

(or $d\dot{X}$ corresponding to $\{X_1 + \mathfrak{m}, ..., X_k + \mathfrak{m}\}$ if we identify $\mathfrak{x} = \mathfrak{m}\backslash\mathfrak{n}$). The partial transforms are

$$\mathscr{F}_1\psi(l', X) = \int_{\mathfrak{m}} \psi(M + X)\, e^{2\pi i\langle l', M\rangle}\, dM, \quad \text{all} \quad X \in \mathfrak{x},\ l' \in \mathfrak{m}^*, \quad \psi \in \mathscr{S}(\mathfrak{m} \times \mathfrak{n}) \qquad (dM = ds)$$

$$\mathscr{F}_2\psi(l'') = \int_{\mathfrak{m}\backslash\mathfrak{n}} \psi(\mathfrak{m} + X)\, e^{2\pi i\langle l'', x\rangle}\, d\dot{X}, \qquad \text{all} \quad l'' \in \mathfrak{x}^*, \qquad \psi \in \mathscr{S}(\mathfrak{m}/\mathfrak{n}) \qquad (d\dot{X} = dt).$$

Clearly, if $l' \in \mathfrak{m}^*$, $l'' \in (\mathfrak{m}/\mathfrak{n})^* = \mathfrak{m}^\perp$, and dW is the Euclidean measure $ds\, dt$ on $\mathfrak{n}$, then $dM\, d\dot{X} = dW$ and

$$\begin{aligned}\mathscr{F}_2\mathscr{F}_2\psi(l', l'') &= \int \left[\int \psi(M + X)\, e^{2\pi i\langle l', M\rangle}\, dM\right] e^{2\pi i\langle l'', X\rangle}\, d\dot{X} \\ &= \int_{\mathfrak{n}} \psi(W)\, e^{2\pi i\langle l_0, W\rangle}\, dW = \psi^\wedge(l_0)\end{aligned}$$

is the usual Fourier transform from $\mathscr{S}(\mathfrak{n})$ to $\mathscr{S}(\mathfrak{n}^*)$ if we take $l_0 \in \mathfrak{n}^*$ to be the functional such that $l_0 \mid \mathfrak{m} = l'$, $l_0 \mid \mathfrak{x} = l'' \mid \mathfrak{x}$. For $\varphi \in \mathscr{S}(N)$ we take the related Fourier transform: $\mathscr{F}\varphi = (\varphi \circ \log)^\wedge$.

As m runs through a set of representatives for $R(l)\backslash M$, $l \cdot m - l$ runs through $\mathfrak{m}^\perp$; let $d\dot{m}$ on $R(l)\backslash M$ be chosen to match dl' on $\mathfrak{m}^\perp$ under this correspondence. For $t \in \mathbf{R}^k$, define p_t: $R(l)\backslash M \to \mathfrak{x}^*$ via $p_t(\dot{m}) = l \cdot m \cdot \gamma(t) \mid \mathfrak{x} = [l \cdot m \cdot \gamma(t) - l \cdot \gamma(t)] \mid \mathfrak{x} + l \cdot \gamma(t) \mid \mathfrak{x}$. The values run through $\mathfrak{m}^\perp = \mathfrak{x}^*$; p_t also carries $d\dot{m}$ to dl' (note that $\mathfrak{m}^\perp$ is N-invariant and $\gamma(t)$ preserves dl'). Thus our integral may be rewritten as integrals along the disjoint fibers $l \cdot M \cdot \gamma(t) = l \cdot \gamma(t) + \mathfrak{m}^\perp$:

$$\begin{aligned}K_\varphi(u, t) &= \int_{R(l)\backslash M} \mathscr{F}_2\mathscr{F}_1(\varphi \circ \delta')(l \cdot m \cdot \gamma(t) \mid \mathfrak{m}, l \cdot m \cdot \gamma(t) \mid \mathfrak{x}) \\ &\quad \times e^{-2\pi i\langle l\cdot m\cdot\gamma(t), X(\tau)+M(\sigma)\rangle}\, d\dot{m} \\ &= \int_{R(l)\backslash M} \mathscr{F}(\varphi \circ \delta)(l \cdot m \cdot \gamma(t))\, e^{-2\pi i\langle l\cdot m\cdot\gamma(t), j(\sigma,\tau)\rangle}\, d\dot{m}. \qquad (13)\end{aligned}$$

If we then define Q on $\Sigma \times \Sigma$ via $Q(\gamma(u), \gamma(t)) = -j(\sigma, \tau)$ the integral is nearly in the desired form. Define Q on $N \times N$ usint the unique splitting $n = mx$ $(m \in M,\ x = \gamma(t) \in \Sigma)$:

$$\begin{aligned}Q(n_1, n_2) &= Q(m_1\gamma(u), m_2\gamma(t)) = Q(\exp(Y_1)\,\gamma(u), \exp(Y_2)\,\gamma(t)) \\ &= Q(\gamma(u), \gamma(t)) + Y_1 \circ \gamma(t)^{-1} - Y_2 \circ \gamma(t)^{-1},\end{aligned}$$

where $Y \cdot n = \mathrm{Ad}(n^{-1})Y$. This is consistent with the definition on $\Sigma \times \Sigma$ and yields the global formula for K_φ,

$$K_\varphi(n_1, n_2) = K_\varphi(m_1\gamma(u), m_2\gamma(t)) = e^{2\pi i(\langle l, Y_1\rangle - \langle l, Y_2\rangle)} K_\varphi(\gamma(u), \gamma(t))$$
$$= \int \mathscr{F}(\varphi \circ \delta)(l \cdot m \cdot n_2)\, e^{2\pi i\langle l\cdot m\cdot n_2, Q(n_1, n_2)\rangle}\, d\dot{m}.$$

Finally, one can verify that $d\dot{m}$ as described above is the same as the canonical choice described in the statement of the theorem, proceding along lines suggested by the dimension in [2, Section 5]; we omit these details. Q.E.D.

Let us now compute $\mathrm{Tr}(\pi_\varphi)$ directly from (12) and (13). Comparison of the results with Kirillov's orbital integral formula yields some curious identities concerning orbit integrals of Fourier transforms. From (13) we get (recall $\mathscr{F}(\varphi \circ \delta) = (\varphi \circ \delta')^\wedge$, where $\delta' = \delta \circ \exp$):

$$\mathrm{Tr}(\pi_\varphi) = \int_{\mathbf{R}^k} K_\varphi(\gamma(t), \gamma(t))\, dt = \int_{\mathbf{R}^k} \int_{R_l\backslash M} \mathscr{F}(\varphi \circ \delta)(l \cdot m \cdot \gamma(t))\, d\dot{m}\, dt$$
$$= \int_{M\backslash N} \int_{R_l\backslash M} (\varphi \circ \delta')^\wedge(l \cdot m \cdot n)\, d\dot{m}\, d\dot{n}$$
$$= \int_{\mathscr{O}} (\varphi \circ \delta')^\wedge(l')\, d\mu_{\mathscr{O}}(l'), \qquad \text{all} \quad \varphi \in \mathscr{S}(N), \tag{14}$$

since $d\dot{m}\, d\dot{n} = d\mu_{\mathscr{O}}$. Starting from (10), we have $\xi(t, t) \equiv \eta(t, t) \equiv 0$, so that computations similar to those above yield

$$\mathrm{Tr}(\pi_\varphi) = \int_{\mathbf{R}^k} \mathscr{F}_1(\varphi \circ \beta)(l \cdot \gamma(t) \mid \mathfrak{m}, 0)\, dt$$
$$= \int_{\mathfrak{m}^\perp} \int_{\mathbf{R}^k} \mathscr{F}_2\mathscr{F}_1(\varphi \circ \beta)(l \cdot \gamma(t) \mid \mathfrak{m}, l')\, dt\, dl'$$
$$= \cdots = \int_{R_l\backslash M} \int_{\mathbf{R}^k} (\varphi \circ \beta')^\wedge(l \cdot m \cdot \gamma(t))\, d\dot{m}\, dt$$
$$= \int_{\mathscr{O}} (\varphi \circ \beta')^\wedge(l')\, d\mu_{\mathscr{O}}(l'), \qquad \text{all} \quad \varphi \in \mathscr{S}(N) \tag{15}$$

where $\beta' = \beta \circ j^{-1}$: $\mathfrak{n} \to N$. Of course we also have Kirillov's formula :

$$\mathrm{Tr}(\pi_\varphi) = \int (\varphi \circ \exp)^\wedge(l)\, d\mu_{\mathscr{O}}(l'), \qquad \text{all} \quad \varphi \in \mathscr{S}(N). \tag{16}$$

Formulas (14)–(16) show that for any $\varphi \in \mathscr{S}(N)$ the integral of $\psi^\wedge$, $\psi = \varphi \circ \exp \in \mathscr{S}(\mathfrak{n})$, over the orbit $\mathscr{O}$ is unaffected if ψ is first scrambled by composition with one of the nonlinear operators $\delta'' = \log \circ \delta'$ or $\beta'' = \log \circ \beta'$ on $\mathfrak{n}$. There are many such operators, corresponding to possible choices of $\mathfrak{m}$ and a weak Malcev basis through $\mathfrak{m}$. This invariance is quite distinct from the $\mathrm{Ad}^*(N)$-invariance of orbital integrals, and is a kind of non-linear Poisson summation formula. If there is a single $\mathfrak{m} \lhd \mathfrak{n}$ maximal subordinate for all $l \in \mathfrak{n}^*$ in general position, if $d\lambda$ is the Plancherel measure on generic orbits $\{\mathcal{O}(\lambda) : \lambda \in \Delta\}$, and if $F(l)$ is any bounded measurable function on $\mathfrak{n}^*$ constant on orbits in Δ, then $\int^{\oplus} \mu_{\mathcal{O}(\lambda)}\, d\lambda$ is just Euclidean measure dl on $\mathfrak{n}^*$ and

$$\int_{\mathfrak{n}^*} (\psi \circ \delta'')^\wedge(l) F(l)\, dl = \int (\psi \circ \beta'')^\wedge(l) F(l)\, dl = \int \psi^\wedge(l) F(l)\, dl$$

for all $\psi \in \mathscr{S}(\mathfrak{n})$. If $F \equiv 1$ this reduces to $\int_{\mathfrak{n}^*} (\psi \circ \delta'')^\wedge\, dl = \cdots = \int \psi^\wedge(l)\, dl$ which is a classical Poisson summation: the integrals are just the values $\psi \circ \delta''(0), \ldots, \psi(0)$ which are equal since $\delta''(0) = \beta''(0) = 0$, by definition.

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